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CRITICAL EXPONENT AND CRITICAL BLOW-UP FOR QUASILINEAR PARABOLIC EQUATIONS

BY

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ABSTRACT

We consider nonnegative solutions to the Cauchy problem or to the exterior Dirichlet problem for the quasilinear parabolic equations $u_t = \Delta u^m + u^p$ with $1 < m < p$. In case of the Cauchy problem, it is well known that $p_m^* = m + 2/N$ is the critical exponent of blow-up. Namely, if $p < p_m^*$, then all nontrivial solutions blow up in finite time (blow-up case), and if $p > p_m^*$, then there are nontrivial global solutions (global existence case). In this paper we show: (i) For the Cauchy problem, p_m^* belongs to the blow-up case. (ii) For the exterior Dirichlet problem, p_m^* also gives the critical exponent of blow-up.

1. Introduction

Let $\Omega = \mathbf{R}^N$ or an exterior domain in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$. We consider the initial-boundary value problem

$$\partial_t u = \Delta u^m + u^p, \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \tag{1.3}$$

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(if $\Omega = \mathbf{R}^N$, the boundary condition (1.3) is dropped), where $m \geq 1$, $p > 1$ and $u_0(x) \geq 0$. We shall only consider nonnegative solutions u. If $u_0(x) \in BC(\overline{\Omega})$ (bounded continuous functions), then a unique *weak* nonnegative solution of (1.1)–(1.3) exists at least for sufficiently small $T > 0$. When $T = \infty$ we say u is global. Otherwise we say u blows up in finite time. In the latter case, it follows that

$$\sup_{x \in \mathbf{R}^N} u(x,t) \to \infty \quad \text{as} \quad t \uparrow T.$$

In case $\Omega = \mathbf{R}^N$, the following results are well known by the classical papers of Fujita [3] (when $m = 1$) and Galaktionov et al. [5] (when $m > 1$).

(I) If $1 < p < m + 2/N$, then all nontrivial nonnegative solutions of (1.1)–(1.3) blow up in finite time.

(II) If $p > m + 2/N$, then global solutions of (1.1)–(1.3) exist when the initial data are sufficiently small. Case (I) is called the blow-up case; (II) is called the global existence case. The number

$$p_m^* = m + 2/N \tag{1.4}$$

is called the critical exponent. Hayakawa [7] and Weissler [19] completed Fujita's result by showing that p_1^* belongs to the blow-up case. More recently, Bandle–Levine [1] proved that p_1^* is still the critical exponent if $\mathbf{R}^N$ is replaced by any exterior domain Ω. However, in this case it is not yet established whether or not p_1^* is in the blow-up case. On the other hand, in case $m > 1$, there are few works which shapen or extend the above result of [5], and p_m^* is not known to be in the blow-up case or not even when $\Omega = \mathbf{R}^N$ (a rather complete survey of such results and related problems was given in Levine [11]). It is the purpose of this paper to answer some of these open problems. More precisely, our results are summarized in the following two theorems.

THEOREM 1: *The case $\Omega = \mathbf{R}^N$. If $N \geq 1$ and $1 \leq m < p \leq p_m^*$, then all nonnegative, nontrivial solutions of the Cauchy problem* (1.1), (1.2) *blow up in finite time.*

THEOREM 2: *The case $\Omega \neq \mathbf{R}^N$.*

(i) *If $N \geq 2$ and $1 \leq m < p < p_m^*$, then all nonnegative, nontrivial solutions of* (1.1)–(1.3) *blow up in finite time.*

(ii) *If $N \geq 2$ and $p > p_m^*$ $(m \geq 1)$, then global solutions of* (1.1)–(1.3) *exist when the initial data are sufficiently small.*

As is mentioned above, these theorems involve some known results. However, in the following, we shall give a unified proof to all the assertions of Theorems 1 and 2 (i). Our proof restricted to the known results differs from those given in the preceding works. Theorem 2 (ii) easily follows from the corresponding result (II) for $\Omega = \mathbf{R}^N$ if we make use of a comparison theorem for parabolic equations. Moreover, the comparison theorem is applicable to reduce Theorem 2 (i) to the special problem (1.1)–(1.3) with $\Omega = E_R \equiv \{x : |x| > R\}$, $R > 0$. In the following, without loss of generality, we assume that $u_0(x)$ is of compact support in Ω:

$$u_0(x) \in C_0(\Omega). \tag{1.5}$$

Then it follows that $u(\cdot, t) \in L^1(\Omega)$ for each $t \in (0, T)$ (Proposition 2.2). To show Theorems 1 and 2 (i) with $\Omega = E_R$, we shall first establish a blow-up condition on the initial data. Let u be a global solution to (1.1)–(1.3) with $\Omega = E_R$ (we mean $E_0 = \mathbf{R}^N$). Put

$$I_R(t;\epsilon) = \int_{E_R} u(x,t)\rho_R(|x|)e^{-\epsilon(|x|-R)^2}dx, \quad \epsilon \in (0,1), \tag{1.6}$$

where $\rho_0(r) \equiv 1$ and, if $R > 0$,

$$\rho_R(r) = \begin{cases} \frac{r-R}{r}, & N \geq 3, \\ \log r - \log R, & N = 2. \end{cases} \tag{1.7}$$

Then our blow-up condition in turn implies that

$$I_0(t;\epsilon) \leq C_0(N)\epsilon^{-N/2+1/(p-m)} \tag{1.8}$$

for any $t \geq 0$, where $C_0(N) = \pi^{N/2}(2N)^{1/(p-m)}$, and if $R > 0$,

$$I_R(t;\epsilon) \leq \begin{cases} C_R(N)\epsilon^{-N/2+1/(p-m)}\{1+o(1)\}, & N \geq 3 \\ C_R(2)\epsilon^{-1+1/(p-m)}\log(\epsilon^{-1/2})\{1+o(1)\}, & N = 2 \end{cases} \tag{1.9}$$

for any $t \geq 0$, where $C_R(N) = \pi^{N/2}(2N+4)^{1/(p-m)}$.

If $m < p < p_m^*$, then letting $\epsilon \to 0$ in (1.8) and (1.9), we obtain

$$\int_{E_R} u(x,t)\rho_R(|x|)dx = 0, \tag{1.10}$$

which concludes $u(x,t) \equiv 0$. On the other hand, if $p = p_m^*$, then except the case $N = 2$ and $R > 0$, we can let $\epsilon \to 0$ in (1.8) and (1.9) to obtain

$$\int_{E_R} u(x,t)\rho_R(|x|)dx \le C_R(N). \tag{1.11}$$

This and equation (1.1) imply another inequality

$$\int_0^\infty \int_{E_R} u(x,t)^p \rho_R(|x|)dxdt \le C_R(N). \tag{1.12}$$

If we restrict ourselves to the case $R = 0$, i.e., $\Omega = \mathbf{R}^N$, then with this inequality, $u(x,t) \equiv 0$ is also concluded by reduction to absurdum.

The blow-up condition which implies (1.8) or (1.9) is obtained based on an extended form of Jensen's inequality. Similar forms have been introduced and used in Bandle–Levine [1] when $m = 1$. In the proof of critical blow-up, Hayakawa [7] and Weissler [19] reduced the problem to an integral equation, and made use of the explicit formula of the heat kernel. Similar approaches were employed by Levine–Meier [12] and Hamada [6] to establish another critical blow-up in cones. Our treatment differs from theirs since ours is based just on the weak formulation of equation (1.1) which we use to obtain inequality (1.12).

The rest of the paper is organized as follows: In the next §2 we define a weak solution of (1.1), and prepare several preliminery propositions. A blow-up condition on initial data is given in Proposition 2.3. In §3 we consider the case $\Omega = \mathbf{R}^N$ and prove Theorem 1. Finally, in §4 we partly extend the argument of §3 to the exterior problem (1.1)–(1.3). Theorem 2 (i) is initially proved for the special case $\Omega = E_R$. To complete the proof of the general case, we are able to apply the comparison theorem. Theorem 2 (ii) also applies the comparison theorem.

After completing this work, we encountered the works of Galaktionov [4] and Kawanago [9] which also prove our Theorem 1. Their proofs are different from ours. Galaktionov [4] reduces the problem to the nonexistence problem of stationary profile for some rescaled equation to (1.1). Kawanago [9] uses a blow-up condition given by Levine–Sacks [13] to prove his results. This condition differs from ours. More recently, based on our results, some progress has been made in the related problems. Suzuki [18] proved that the critical exponent p_m^* in Theorem 2 also belongs to the blow-up case if $N \ge 3$. Mochizuki–Mukai [14] extended our results and Suzuki's to the fast diffusion problem (1.1)–(1.3) with $\max\{0, 1-2/N\} < m < 1$.

2. Preliminaries

We begin with the definition of a weak solution of (1.1).

Definition: By a solution of equation (1.1) in $\Omega \times (0,T)$ we mean a function $u(x,t)$ in $\overline{\Omega} \times [0,T)$ such that

(i) $u(x,t) \geq 0$ and $\in BC(\overline{\Omega} \times [0,T'])$ for any $0 < T' < T$.

(ii) For any bounded domain $G \subset \Omega$, $0 \leq \tau < T$ and nonnegative $\varphi(x,t) \in C^2(\overline{\Omega} \times [0,T))$ which vanishes on the boundary ∂G,

$$\int_G u(x,\tau)\varphi(x,\tau)dx - \int_G u(x,0)\varphi(x,0)dx$$
$$= \int_0^\tau \int_G \{u\varphi_t + u^m \Delta\varphi + u^p\varphi\}dxdt - \int_0^\tau \int_{\partial G} u^m \partial_n \varphi dSdt, \tag{2.1}$$

where n denotes the outer unit normal to the boundary. A supersolution [or subsolution] is similarly defined with equality (2.1) replaced by $\geq$ [or $\leq$].

Note that every weak solution is classical near the point (x,t) where $u(x,t) > 0$. Assume that

$$\int_\Omega \{\varphi + |\partial_t\varphi| + |\nabla\varphi| + |\Delta\varphi|\}dx < \infty \quad \text{and} \quad \varphi(x,t) = 0 \text{ on } \partial\Omega \times [0,T). \tag{2.2}$$

Then by a limit procedure we easily have

$$\int_\Omega u(x,\tau)\varphi(x,\tau)dx - \int_\Omega u(x,0)\varphi(x,0)dx$$
$$= \int_0^\tau \int_\Omega \{u\varphi_t + u^m \Delta\varphi + u^p\varphi\}dxdt - \int_0^\tau \int_{\partial\Omega} u^m \partial_n \varphi dSdt. \tag{2.3}$$

The following comparison principle is well known for quasilinear equation (1.1) (see, e.g., Bertsch–Kersner–Peletier [2; Appendix]). The result will be freely used in the sequel.

PROPOSITION 2.1: *Let u [or v] be a supersolution [or subsolution] of* (1.1). *If $u \geq v$ on the parabolic boundary of $\Omega \times (0,T)$, then we have $u \geq v$ in the whole $\overline{\Omega} \times [0,T)$.*

As a result of this comparison principle we can show the following

PROPOSITION 2.2: *Let $u(x,t)$ be the solution to (1.1)–(1.3) with $u_0(x) \in C_0(\Omega)$. Then $u(\cdot,t) \in L^1(\Omega)$ for each $0 \le t < T$. More precisely, for any fixed $0 < T' < T$, we have:*

(i) *If $m = 1$, then there exists a $C(T') > 0$ such that*

$$u(x,t) \le C(T')e^{-|x|^2/4} \quad \text{in } \Omega \times [0,T']. \tag{2.4}$$

(ii) *If $m > 1$, then there exists a compact domain $K \subset \overline{\Omega}$ such that*

$$\operatorname{supp} u(\cdot,t) \subset K \quad \text{for any } t \in [0,T'], \tag{2.5}$$

where $\operatorname{supp} u$ is the support of u in $x \in \Omega$.

Proof: (i) Put

$$a(T') = \sup_{(x,t)\in\Omega\times(0,T')} u(x,t)^{p-1}.$$

Then it follows from (1.1) that

$$\partial_t u \le \Delta u + a(T')u, \quad (x,t) \in \Omega \times (0,T'].$$

Let $M > 0$ be chosen to satisfy

$$u_0(x) \le M(4\pi)^{-N/2}e^{-|x|^2/4}, \quad x \in \Omega.$$

Then comparing $e^{-a(T')t}u(x,t)$ with the solution to the initial value problem

$$\begin{cases} \partial_t w = \Delta w, & (x,t) \in \mathbf{R}^N \times (0,\infty), \\ w(x,0) = M(4\pi)^{-N/2}e^{-|x|^2/4}, & x \in \mathbf{R}^N, \end{cases}$$

we obtain

$$u(x,t) \le e^{a(T')t}M\big(4\pi(t+1)\big)^{-N/2}e^{-|x|^2/4(t+1)}$$

in $\Omega \times [0,T']$. Thus, (i) is proved.

(ii) Let $a(T')$ be as above. As is easily seen, the initial value problem

$$\begin{cases} \partial_t w = \Delta w^m + a(T')w, & (x,t) \in \mathbf{R}^N \times (0,\infty) \\ w(x,0) = u_0(x), & x \in \mathbf{R}^N \end{cases}$$

has a global solution w. In fact, this problem has a supersolution

$$\overline{w}(t) = \overline{u}_0 e^{a(T')t}, \quad \overline{u}_0 = \sup_{x\in\Omega} u_0(x)$$

which is global in $t \geq 0$. By the comparison argument we have $u \leq w \leq \overline{w}$ in $\Omega \times (0, T']$. On the other hand, as is proved in Mochizuki–Suzuki [15; Theorem 2.4], $w(\cdot, t)$ is compactly supported in $\mathbf{R}^N$ for each $t \in [0, T']$. Thus, (ii) is proved. ∎

Next, we shall establish a blow-up condition on initial data. The proof is almost the same as in Imai–Mochizuki [8; Theorem 1.1]. In case of $m = 1$, a similar condition has been used in Bandle–Levine [1; Theorem 2.3].

We assume the existence of the pair $\{\lambda, s(x)\}$ satisfying the following properties:

$$\begin{gathered} \lambda \geq 0; \quad s(x) \in C^2(\overline{\Omega}) \quad \text{and} \quad s(x) > 0 \text{ in } \Omega; \\ \Delta s(x) \geq -\lambda s(x), \quad \text{in } \Omega \quad \text{and} \quad s(x) = 0 \quad \text{on } \partial\Omega; \\ \int_\Omega \{s(x) + |\nabla s(x)| + |\Delta s(x)|\} dx < \infty. \end{gathered} \tag{2.6}$$

With this function $s(x)$ we define $J(t)$, $t \geq 0$, as follows:

$$J(t) = \left(\int_\Omega s(x) dx\right)^{-1} \int_\Omega u(x,t) s(x) dx. \tag{2.7}$$

PROPOSITION 2.3: *Let u be a solution to (1.1)–(1.3) with $p > m$. If u_0 is large enough to satisfy*

$$J(0) > \lambda^{1/(p-m)}, \tag{2.8}$$

then u is never global in t.

Proof: We choose $\varphi = s(x)$ in (2.3). Then noting (2.6), we have

$$J(\tau) - J(0) \geq \left(\int_\Omega s(x) dx\right)^{-1} \int_0^\tau \int_\Omega \{-\lambda u^m + u^p\} s(x) dx dt.$$

Put

$$\Gamma(\xi) = \begin{cases} -\lambda \xi^m + \xi^p, & \xi \geq \left(\dfrac{m\lambda}{p}\right)^{1/(p-m)}, \\ -\dfrac{p-m}{m}\left(\dfrac{m\lambda}{p}\right)^{p/(p-m)}, & 0 \leq \xi < \left(\dfrac{m\lambda}{p}\right)^{1/(p-m)}. \end{cases} \tag{2.9}$$

Then, by the Jensen inequality,

$$J(\tau) \geq J(0) + \int_0^\tau \Gamma(J(t)) dt \equiv \alpha(\tau). \tag{2.10}$$

Since $\Gamma(\xi) > 0$ in $\xi > \lambda^{1/(p-m)}$, the continuity of $J(t)$ in t and (2.8) imply $J(t) > \lambda^{1/(p-m)}$ in $0 \le t < t_0$ if $t_0 > 0$ is chosen sufficiently small. We shall show $J(t) > \lambda^{1/(p-m)}$ as long as $J(t)$ exists. Assume on the contrary that $J(t) \le \lambda^{1/(p-m)}$ for some $t \ge t_0$. Let τ_0 be the smallest such value. Inequality (2.10) with $\tau = \tau_0$ then leads to a contradiction since we have $\int_0^{\tau_0} \Gamma(J(t))dt > 0$. The above result shows that $\alpha(\tau) \ge J(0)$ in any τ. Moreover, since $\Gamma(\xi)$ is increasing in $\xi > \lambda^{1/(p-m)}$, it follows from (2.10) that

$$\Gamma\big(J(\tau)\big) \ge \Gamma\big(\alpha(\tau)\big) > 0.$$

Integrate both sides of the inequality $1 \le \Gamma\big(J(\tau)\big)/\Gamma\big(\alpha(\tau)\big)$ over $(0, t)$. Then we have

$$t \le \int_{J(0)}^{\alpha(t)} \frac{d\xi}{\Gamma(\xi)} \le \int_{J(0)}^{\infty} \frac{d\xi}{-\lambda\xi^m + \xi^p} < \infty,$$

from which we see that u is never global in t. ∎

The weak local solution u to (1.1)–(1.3) has been constructed as in Oleinik et al. [16] based on a result of Ladyzhenskaja et al. [10] for nondegenerate problems. This shows that the existence time of $u(t)$ depends only on the value

$$\|u_0\|_\infty = \sup_{x \in \mathbf{R}^N} u_0(x).$$

In fact, by means of the uniqueness of solutions and the comparison principle, $u(t)$ is shown to exist at least up to the existence time of the supersolution

$$v(t) = \left\{ \|u_0\|_\infty^{-p+1} - (p-1)t \right\}^{-1/(p-1)}$$

to (1.1)–(1.3). The following proposition is a result of this fact.

Proposition 2.4: *Let u be the solution to* (1.1)–(1.3). *If u is not global in t, then it blows up in finite time. More precisely, let $T_1 > 0$ be the maximal existence time of u. Then we have*

$$\sup_{x \in \mathbf{R}^N} u(x,t) \to \infty \quad \text{as } t \uparrow T_1. \tag{2.11}$$

We shall close this section by giving a concrete expression of the (elementary) solution to the initial value problem

$$\begin{cases} \partial_t v = \Delta v^m, & (x,t) \in \mathbf{R}^N \times (0,\infty), \\ v(x,0) = L\delta(x), & x \in \mathbf{R}^N, \end{cases} \tag{2.12}$$

where $L > 0$ and $\delta(x)$ is Dirac's δ-function.

PROPOSITION 2.5: *Let*

$$\ell = \left(m - 1 + \frac{2}{N}\right)^{-1} = (p_m^* - 1)^{-1} \tag{2.13}$$

and

$$G_m(s) = \begin{cases} (4\pi)^{-N/2} e^{-s^2/4}, & m = 1, \\ \left[A - Bs^2\right]_+^{1/(m-1)}, & m > 1, \end{cases} \tag{2.14}$$

where $[a]_+ = \max\{a, 0\}$, $B = \dfrac{(m-1)\ell}{2mN}$ *and* $A > 0$ *is chosen to satisfy*

$$\int_{\mathbf{R}^N} G_m(|x|)dx = 1.$$

Then the solution to (2.12) *is given by*

$$E_m(x, t; L) = L\left(L^{m-1}t\right)^{-\ell} G_m\left(|x|(L^{m-1}t)^{-\ell/N}\right). \tag{2.15}$$

Proof: If $m = 1$, (2.15) gives the usual heat kernel. (2.15) with $m > 1$ is also well known as the Barenblatt solution to the porous media equation (2.12) (see, e.g., Pattle [17]). ∎

3. Proof of Theorem 1

In this section we consider the solution $u(x, t)$ to the Cauchy problem (1.1), (1.2) with $\Omega = \mathbf{R}^N$ and $u_0(x) \in C_0(\mathbf{R}^N)$.

Theorem 1 will be proved in a series of lemmas.

LEMMA 3.1: *Assume that u is global. Then we have*

$$I_0(t; \epsilon) \equiv \int_{\mathbf{R}^N} u(x, t) e^{-\epsilon|x|^2} dx \le C_0(N) \epsilon^{-N/2 + 1/(p-m)}, \tag{3.1}$$

for any $t \ge 0$, where $C_0(N) = \pi^{N/2}(2N)^{1/(p-m)}$.

Proof: Choose $s(x) = s_\epsilon(x) = e^{-\epsilon|x|^2}$ in (2.7). As is easily verified, this function satisfies properties (2.6) with $\Omega = \mathbf{R}^N$ and $\lambda = 2N\epsilon$. Since

$$\int_{\mathbf{R}^N} e^{-\epsilon|x|^2} dx = \epsilon^{-N/2} \int_{\mathbf{R}^N} e^{-|y|^2} dy = \pi^{N/2} \epsilon^{-N/2},$$

The blow-up condition (2.8) is reduced to

$$\int_{\mathbf{R}^N} u_0(x) e^{-\epsilon|x|^2} dx > \pi^{N/2} \epsilon^{-N/2} (2N\epsilon)^{1/(p-m)} = C_0(N) \epsilon^{-N/2 + 1/(p-m)}.$$

Thus, every global solution u must satisfy the converse inequality (3.1). ∎

LEMMA 3.2: *Assume that u is global.*

(i) *If $1 \le m < p < p_m^*$, then we have*

$$\int_{\mathbf{R}^N} u(x,t)dx = 0 \quad \text{for any } t \ge 0. \tag{3.2}$$

(ii) *If $p = p_m^*$, then we have*

$$\int_{\mathbf{R}^N} u(x,t)dx \le C_0(N) = (2N\pi)^{N/2} \quad \text{for any } t \ge 0. \tag{3.3}$$

Proof: Since $u(\cdot,t) \in L^1(\mathbf{R}^N)$ by Proposition 2.2, the Lebesgue dominated convergence theorem shows

$$\lim_{\epsilon \to 0} \int_{\mathbf{R}^N} u(x,t)e^{-\epsilon|x|^2}dx = \int_{\mathbf{R}^N} u(x,t)dx.$$

On the other hand, in (3.1) the exponent of ϵ is

$$-\frac{N}{2} + \frac{1}{p-m} = \frac{N(p_m^* - p)}{2(p-m)} \ge 0.$$

Thus, letting $\epsilon \to \infty$ in (3.1), we conclude the assertions of the lemma. ∎

LEMMA 3.3: *Assume that u is global. If $p = p_m^*$, then we have*

$$\int_0^\infty \int_{\mathbf{R}^N} u(x,t)^p dx \le (2N\pi)^{N/2}. \tag{3.4}$$

Proof: We also use Proposition 2.2. Since $u(\cdot,t) \in L^1(\mathbf{R}^N)$, we can choose $\varphi \equiv 1$ as a test function in (2.3). Then we have

$$\int_{\mathbf{R}^N} u(x,\tau)dx - \int_{\mathbf{R}^N} u_0(x)dx = \int_0^\tau \int_{\mathbf{R}^N} u(x,t)^p dxdt, \tag{3.5}$$

and (3.4) is concluded from (3.3) and (3.5). ∎

The next lemma will estimate the solution v to the Cauchy problem

$$\begin{cases} \partial_t v = \Delta v^m, & (x,t) \in \mathbf{R}^N \times (0,\infty) \\ v(x,0) = u_0(x), & x \in \mathbf{R}^N \end{cases} \tag{3.6}$$

from below.

LEMMA 3.4: *Let* $u_0(x) \not\equiv 0$.

(i) *If* $m = 1$, *then for any* $t_0 > 0$, *there exists a constant* $C(t_0) > 0$ *such that*

$$v(x,t) \geq C(t_0)E_1(x,t/2;1) \quad \text{in } \mathbf{R}^N \times [t_0,\infty). \tag{3.7}$$

(ii) *If* $m > 1$, *then there exists* $L_1 > 0$ *and* $t_1 > 0$ *such that*

$$v(x,t) \geq E_m(x,t+t_1;L_1) \quad \text{in } \mathbf{R}^N \times [0,\infty). \tag{3.8}$$

Proof: Without loss of generality, we can assume that $u_0(0) > 0$. Then there exists a small $\delta > 0$ such that $u_0(x) \geq a > 0$ in $B_\delta = \{x; |x| < \delta\}$.

(i) We have

$$v(x,t) = E_1(\cdot,t;1) * u_0(x) \geq (4\pi t)^{-N/2} e^{-|x|^2/2t} \int_{\mathbf{R}^N} e^{-|y|^2/2t} u_0(y)dy,$$

where $*$ means the convolution in $\mathbf{R}^N$. Then choosing

$$C(t_0) = 2^{-N/2} a \int_{B_\delta} e^{-|y|^2/2t_0} dy > 0,$$

we obtain (3.7).

(ii) Note that (2.15) with $m > 1$ gives

$$E_m(x,t;L) = L^{2\ell/N} t^{-\ell} \left[A - B|x|^2 (L^{m-1}t)^{-2\ell/N} \right]_+^{1/(m-1)}.$$

For fixed $L > 0$ we first choose $t_1 > 0$ so small that $(A/B)^{1/2}(L^{m-1}t_1)^{\ell/N} < \delta$. Next, we choose $L_1 > 0$ very small to satisfy $L_1^{2\ell/N} t_1^{-\ell} A < a$. Then we have

$$u_0(x) \geq E_m(x,t_1;L_1).$$

Applying the comparison argument, we obtain (3.8). ∎

Proof of Theorem 1: As is mentioned in Proposition 2.4, every nonglobal solution to (1.1), (1.2) blows up in finite time. So, to complete the proof we have only to show that every nontrivial solution is nonglobal. Contrary to the conclusion, assume that for given $u_0(x) \not\equiv 0$, the Cauchy problem (1.1), (1.2) has a nontrivial global solution u.

In case $1 \leq m < p < p_m^*$, this assumption and Lemma 3.2 (i) contradict each other since (3.2) implies $u(x,t) \equiv 0$. So, we have only to consider the critical case $p = p_m^*$.

Let v satisfy (3.6). Then v becomes a subsolution to (1.1), (1.2), and we have $v \le u$ in the whole $\mathbf{R}^N \times [0,\infty)$. Combining this and Lemma 3.4, we obtain

$$u(x,t) \ge \begin{cases} C(t_0)E_1(x,t/2;1), & t \ge t_0, \quad m=1, \\ E_m(x,t+t_1;L_1), & t \ge 0, \quad m>1. \end{cases}$$

Thus, it follows from Lemma 3.3 that

$$\begin{aligned}(2N\pi)^{N/2} &\ge \int_{t_0}^{\tau}\int_{\mathbf{R}^N} \{C(t_0)E_1(x,t/2;1)\}^p dxdt \\ &= C(t_0)^p \int_{t_0}^{\tau} (2\pi t)^{-Np/2}(2t)^{N/2}dt \int_{\mathbf{R}^N} e^{-p|z|^2}dz \end{aligned} \tag{3.9}$$

for any $\tau \ge t_0$ if $m=1$, and

$$\begin{aligned}(2N\pi)^{N/2} &\ge \int_0^{\tau}\int_{\mathbf{R}^N} E_m(x,t+t_1;L_1)^p dxdt \\ &= L_1^p \int_{t_1}^{\tau} (L_1^{m-1}t)^{-p\ell}(L_1^{m-1}t)^{\ell}dt \int_{\mathbf{R}^N} G_m(|x|)^p dx \end{aligned} \tag{3.10}$$

for any $\tau \ge t_1$ if $m>1$. Here, since $p=p_m^*$, we have $N(p-1)/2=1$ in (3.9) and $\ell(p-1)=1$ in (3.10). Then the right sides of both inequalities go to ∞ if we let $\tau \to \infty$.

Thus, a contradiction occurs, and the proof of Theorem 1 is completed. ∎

4. Proof of Theorem 2

We shall first consider the special case $\Omega = E_R$, $R>0$. In order to obtain an inequality corresponding to (3.1), we have to modify the function $e^{-\epsilon|x|^2}$ to fit the exterior problem.

LEMMA 4.1: *We put $s_\epsilon(x) = \rho_R(|x|)e^{-\epsilon(|x|-R)^2}$, where*

$$\rho_R(r) = \begin{cases} \dfrac{r-R}{r}, & N \ge 3, \\ \log r - \log R, & N=2. \end{cases} \tag{4.1}$$

Then this $s_\epsilon(x)$ satisfies properties (2.6) with $\Omega = E_R$ and $\lambda = 2(N+2)\epsilon$.

Proof: We shall show the inequality

$$\Delta s_\epsilon(x) \ge -2(N+2)\epsilon s_\epsilon(x) \quad \text{in } E_R. \tag{4.2}$$

The other properties are more easily verified.

We put $r = |x|$. Then $\partial_r \rho_R = Rr^{-2}$ if $N \geq 3$ and $= r^{-1}$ if $N = 2$, and

$$\Delta\rho_R = \Big(\partial_r^2 + \frac{N-1}{r}\partial_r\Big)\rho_R = \begin{cases} (N-3)Rr^{-3}, & N \geq 3, \\ 0, & N = 2. \end{cases}$$

Noting $\Delta\rho_R \geq 0$, we have

$$\begin{aligned} \Delta s_\epsilon(x) &\geq \rho_R \Delta e^{-\epsilon(r-R)^2} + 2(\partial_r \rho_R)\partial_r e^{-\epsilon(r-R)^2} \\ &\geq -2N\epsilon s_\epsilon(x) - 4\epsilon(r-R)(\partial_r\rho_R)e^{-\epsilon(r-R)^2}. \end{aligned} \tag{4.3}$$

Here

$$(r-R)\partial_r\rho_R = \begin{cases} \dfrac{R}{r}\rho_R, & N \geq 3, \\ \dfrac{r-R}{r(\log r - \log R)}\rho_R, & N = 2. \end{cases} \tag{4.4}$$

Since $\dfrac{r-R}{r(\log r - \log R)} \leq \dfrac{R}{r} \leq 1$ in E_R, (4.3) and (4.4) show (4.2). ∎

LEMMA 4.2: *For $s_\epsilon(x)$ given above, we have*

$$\int_{E_R} s_\epsilon(x)dx = \begin{cases} \pi^{N/2}\epsilon^{-N/2}\{1+o(1)\}, & N \geq 3 \\ \pi\epsilon^{-1}\log(\epsilon^{-1/2})\{1+o(1)\}, & N = 2 \end{cases} \tag{4.5}$$

as $\epsilon \to 0$.

Proof: Note that

$$\int_{E_R} s_\epsilon(x)dx = \epsilon^{-N/2}\omega_N \int_0^\infty (\xi + \epsilon^{1/2}R)^{N-1}\rho_R(\epsilon^{-1/2}\xi + R)e^{-\xi^2}d\xi$$

where ω_N is the surface area of the unit sphere. Here,

$$(\xi + \epsilon^{1/2}R)^{N-1}\rho_R(\epsilon^{-1/2}\xi + R) \leq (\xi + R)^{N-2}\xi, \quad \text{and} \ \to \xi^{N-1} \quad \text{as } \epsilon \to 0$$

if $N \geq 3$, and

$$\begin{aligned} &\big[\log(\epsilon^{-1/2})\big]^{-1}(\xi + \epsilon^{1/2}R)\rho_R(\epsilon^{-1/2}\xi + R) \\ &\leq (\xi + R)\Big\{1 + \log\Big(\frac{\xi + R}{R}\Big)\Big\}, \quad \text{and} \ \to \xi \quad \text{as } \epsilon \to 0 \end{aligned}$$

if $N = 2$. Thus, we conclude the above assertion in virtue of the Lebesgue dominated convergence theorem. ∎

With these two lemmas, we can follow the line of proof of Lemma 3.1 to obtain

LEMMA 4.3: *Let u_R be a global solution to (1.1)–(1.3) with $\Omega = E_R$. Then we have*

$$I_R(t;\epsilon) \equiv \int_{E_R} u_R(x,t)\rho_R(|x|)e^{-\epsilon(|x|-R)^2}dx$$

$$\leq \begin{cases} C_R(N)\epsilon^{-N/2+1/(p-m)}\{1+o(1)\}, & N \geq 3 \\ C_R(2)\epsilon^{-1+1/(p-m)}\log(\epsilon^{-1/2})\{1+o(1)\}, & N = 2 \end{cases} \tag{4.6}$$

for any $t \geq 0$, where $C_R(N) = \pi^{N/2}(2N+4)^{1/(p-m)}$.

Proof of Theorem 2 (i): (Special case) Let u_R be as in the above lemma. Note that $u_R(\cdot,t) \in L^1(E_R)$ by Proposition 2.2, and the exponent of ϵ in (4.6) is

$$-\frac{N}{2} + \frac{1}{p-m} = \frac{N(p_m^* - p)}{2(p-m)} > 0.$$

Thus, letting $\epsilon \to 0$ in (4.6), we obtain

$$\int_{E_R} u_R(x,t)\rho_R(|x|)dx = 0 \tag{4.7}$$

for any $t \geq 0$. Since $\rho_R(|x|) > 0$ in E_R, it follows that $u_R(x,t) \equiv 0$.

As in the case of the Cauchy problem, this proves Theorem 2 (i) for $\Omega = E_R$. ∎

Proof of Theorem 2 (i): (General case) Let u be a global solution to (1.1)–(1.3) with any exterior domain Ω. We choose $R > 0$ so large that $E_R \subset \Omega$. Assume that u is nontrivial. Then $u(x,t) > 0$ in $\Omega \times (0,\infty)$ if $m = 1$. On the other hand, if $m > 1$, by use of the Barenblatt solution (2.15), we easily see that supp $u(\cdot,t)$ spread out to the whole Ω as $t \to \infty$. Thus, from the beginning, we can assume that there exists a nonempty domain $K \subset E_R$ such that $u_0(x) > 0$ in K.

Let u_R be the solution to (1.1), (1.2) with $\Omega = E_R$ and with initial condition

$$u_R(x,0) = \varphi_K(x)u_0(x),$$

where $\varphi_K(x) \in C(K)$ is any function satisfying

$$0 < \varphi_K(x) \leq 1 \quad \text{in } K, \quad \text{and} \quad \varphi_K(x) = 0 \quad \text{on } \partial K.$$

Then we can compare u and u_R to obtain

$$u(x,t) \geq u_R(x,t) \quad \text{in } E_R \times [0,\infty). \tag{4.8}$$

However, as we know in the above proof, u_R blows up in finite time. Hence, (4.8) contradicts the assumption that u is global.

Theorem 2 (i) is thus completely proved. ∎

Proof of Theorem 2 (ii): We choose $u_0(x) \in C_0(\Omega)$, $\not\equiv 0$, so small that the Cauchy problem (1.1),(1.2) in $\mathbf{R}^N \times (0,T)$ has a global, nontrivial solution $\widetilde{u}(x,t)$. Let u be a solution to (1.1)–(1.3) with the same initial data u_0. Then we can also compare these two functions to obtain

$$u(x,t) \leq \widetilde{u}(x,t) \quad \text{in } \Omega \times [0,T).$$

If we assume that u blows up in finite time, this raises again a contradiction since $\widetilde{u}$ is global. Theorem 2 (ii) is thus proved. ∎

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